Brane Cosmology From Heterotic String Theory

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We consider brane cosmologies within the context of five-dimensional effective actions with $\mathcal{O}(\alpha')$ higher curvature corrections. The actions are compatible with bulk string amplitude calculations from heterotic string theory. We find wrapped solutions that satisfy the field equations in an *approximate* but acceptable manner given their complexity, where the internal, four-dimensional, scale factor is naturally inflating, having an exponential De-Sitter form. The temporal dependence of the metric components is nontrivial so that this metric cannot be factored as in a conformally flat case. The effective Planck mass is finite and the brane solutions can localize four-dimensional gravity while the four-dimensional gravitational constant varies with time. The Hubble constant can be freely specified through the initial value of the scalar field, to conform with recent data.

KEY WORDS: branes; cosmology.

1. INTRODUCTION

Recent developments in string theory suggest that matter and gauge interactions may be confined on a brane, embedded in a higher dimensional space (bulk), while gravity can propagate into the bulk (for reviews see Brax and van de Bruck, 2003; Dick, 2001; Maartens, in press; Papantonopoulos, 2000; Quevedo, 2002). Within this context several toy models have been constructed to address such issues as the hierarchy and cosmological constant problems (Binetruy *et al.*, 2000; Lukas *et al.*, 1999a,b). In particular, the large hierarchy between the Standard Model and Planck scales could be explained for an observer on a negative tension flat brane, if the extra dimension was taken to be compact (Randall and Sundrum, 1999a,b). The possibility of a large noncompact dimension was realized in Arkadi-Hamed *et al.* (1998) and Antoniadis *et al.* (1998), while it was shown in Randall and Sundrum (1999a,b) that warping of five-dimensional space could lead to localization of gravity on the brane, even though the size of the extra dimension was of infinite proper length.

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A simple, interesting alternative model has been considered in Arkani-Hamed *et al.* (2000), Kachru *et al.* (2000a,b), where a bulk scalar field ϕ is coupled to the brane tension T_{br}. This is the all-loop contribution to the vacuum energy density of the brane, from the Standard Model fields. For the 4D cosmological constant problem considered there, solutions of the field equations were found, which localize gravity, but possess naked singularities at finite proper distance. This proper distance is given by $y_c = 1/\kappa_{(5)}^2 T_{\text{br}}$ where the five and four-dimensional Planck scales $k_{(5)}^2 = M_{(5)}^{-3}, \kappa_{(4)}^2 = M_{(4)}^{-2}$ are related by

$$T_{\text{br}} = \frac{\kappa_{(4)}^2}{\kappa_{(5)}^4} = \frac{M_{(5)}^6}{M_{(4)}^2}. \tag{1}$$

Then if we momentarily identify the 4D cosmological term with the brane tension $\lambda = T_{\text{br}} \sim (1\text{TeV})^4 = (10^3\,\text{GeV})^4 \sim 10^{-64} M_{(4)}^4$ we obtain

$$M_{(5)} \simeq 10^8\,\text{GeV}, \quad y_c \simeq 1\,\text{mm}, \tag{2}$$

which is acceptable by present day experiments.

It was later realized that the bulk action should also contain the Gauss-Bonnet (GB) term

$$\mathcal{L}_{\text{GB}} = R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}, \tag{3}$$

which is the leading quantum gravity correction, and the only to provide second-order field equations. Some of the early works on the GB gravity include Boulware and Deser (1985) and Wheeler (1986), while the corresponding brane cosmology has been studied, among others, in Bento and Bertolami (1989, 1996), Charmousis and Dufaux (2002), Deruelle and Dolezel (2000), Easson (2003), Easson and Brandenberger (1999), Germani and Sopuerta (2002), Gregory and Padilla (2003), Kim *et al.* (2000), Kofinas *et al.* (2003), Lidsey *et al.* (2002), and Nojiri and Odintsov (2000). The corresponding generalized junction conditions appeared in Davis (2003) and Gravanis and Willison (2003).

If we consider the action (Lidsey, in press)

$$S = \int d^5 x \sqrt{|g|} \left[R - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}e^{-\phi}(\nabla\sigma)^2 - \Lambda e^{-2\phi} \right]$$

$$+ \sum_{i=1}^{2} (-1)^i \sqrt{24}\Lambda \int d^4 x \sqrt{|g_i|} e^{-\phi} \tag{4}$$

then Eq. (4) introduces the fact that the bulk cosmological constant couples to the scalar field through the exponential potential term. However, serious arguments were given in Binétruy *et al.* (2002) and Mavromatos and Rizos (2000, 2003), that one must consider a modified action, instead of the usual one used in Bento and Bertolami (1989, 1996), Charmousis and Dufaux (2002), Deruelle and Dolezel

(2000), Easson (2003), Easson and Brandenberger (1999), Germani and Sopuerta (2002), Gregory and Padilla (2003), Kim *et al.* (2000), Kofinas *et al.* (2003), Lidsey *et al.* (2002), and Nojiri and Odintsov (2000) where the GB term is multiplied by a constant. The modified action comes from heterotic string amplitude calculations, truncated to five dimensions (Gross and Sloan, 1987; Metsaev and Tseytlin, 1987). In this action the GB combination enters through a multiple of the exponential of the dilaton field.

To write this action we use the constants $\kappa_{(i)}^2 = 8\pi G_i = M_{(i)}^{2-i}$, $i = 4, 5$, which represent the fundamental, five-dimensional and the effective four-dimensional Planck masses. The bulk cosmological constant has dimensions $\Lambda_{(5)} = [\text{energy}]^2$, defining an inverse length scale squared, effectively an AdS_5 curvature. Also, a brane cosmological constant would have dimensions $\Lambda_{(4)} = [\text{energy}]^2$ and divided both by the respective mass scales $\kappa_{(i)}^2$ would have dimensions similar to the corresponding brane tensions. With this normalization, the action used in Binétruy *et al.* (2003) and Mavromatos and Rizos (2000, 2003) is dimensionless, and can be written as

$$S_1 = \frac{1}{2\kappa_{(5)}^2} \int d^5x \sqrt{|g_{(5)}|} \left\{ R - \zeta(\nabla\phi)^2 + \alpha \, e^{-\zeta\phi} \big[R_{\text{GB}}^2 + c_2 \zeta^2 \, (\nabla\phi)^4 \big] \right\} \quad (5)$$

Here $\alpha = \alpha'/8g_s^2$, with $l_s = \sqrt{\alpha'}$ the string length, and $g_s = \exp(-\phi_0)$ the string coupling constant, where ϕ_0 is the vacuum expectation value of the dilaton field. Also $\zeta = (4/3)$ and $c_2 = (D - 4/D - 2)$ and in our case $c_2 = (1/3)$.

In addition we take as

$$S_2 = \frac{-1}{2\kappa_{(5)}^2} \int d^5x \sqrt{|g_{(5)}|} \left[2\Lambda_{(5)} V(\phi) \right] \quad (6)$$

Using this normalization the bulk potential and also ϕ is dimensionless. In Binétruy *et al.* (2003), it was taken as $V(\phi) = e^{\zeta\phi}$ and this will also be our choice.

So with this normalization the various S_i are dimensionless. We take as

$$S = S_1 + S_2 \quad (7)$$

Solutions to the first part of the action were studied in Binétruy *et al.* (2003), Charmousis *et al.* (2003), Gross and Sloan (1987), Mavromatos and Rizos (2000, 2003), and Metsaev and Tseytlin (1987). However, due to the severe complexity of the equations of motion only metrics of the form

$$ds^2 = e^{2A(y)}\eta_{\mu\nu} \, dx^\mu \, dx^\nu + dy^2 \quad (8)$$

were considered. This choice limits the range of solutions for realistic internal, four-dimensional spacetimes. One of the interesting cases to consider is an inflating internal space. This is because it is generally accepted that the Universe now undergoes accelerated expansion (Kiddle, 2001; Knop *et al.*, in press; Perlmutter *et al.*, 1999; Seto *et al.*, 2001). In this paper we write the field equations for the

action of Eq. (7), and use a metric with internal De-Sitter-like scale factor. The extreme complexity of the field equations does not allow for an exact, mathematical solution. However for a continuous range of one parameter of the model, three of the field equations coincide, with acceptable accuracy, permiting a class of models with interesting characteristics to appear. First the action is a realistic one, coming from string theory. Also the brane, located at $y = 0$, has an inflating internal scale factor. Gravity is localized on the brane, so the effective four-dimensional Planck mass is finite. The class of models does not obey a fine-tuning condition, in the sense that the bulk cosmological constant only controls the temporal evolution of the scalar field. The Hubble constant is determined from the initial value of the scalar field so that it can be adjusted to any value without fine-tuning of the parameters of the action.

2. EQUATIONS OF MOTION

The metric will be of the form

$$ds^2 = -n^2(t, y)\, dt^2 + a^2(t, y) h_{ij}(x)\, dx^i\, dx^j + b^2(t, y)\, dy^2 \qquad (9)$$

The three-metric from Equation (9) is assumed to represent a maximally symmetric space

$$^{(3)}ds^2 = a^2 h_{ij}(x)\, dx^i\, dx^j = a^2 \left[\frac{dr^2}{1 - kr^2} + r^2\, d\Omega_{\mathrm{II}}^2 \right] \qquad (10)$$

with scalar three-curvature $^{(3)}R = 6k/a^2$, $\ k = 0, \pm 1$.
Variation of the action with respect to the five dimensional metric gives:

$$G_{\mu\nu} - \zeta(\nabla_\mu\phi)(\nabla_\nu\phi) + \frac{\zeta}{2} g_{\mu\nu}(\nabla\phi)^2 + g_{\mu\nu}\Lambda_{(5)} V(\phi) + 2\alpha e^{-\zeta\phi} H_{\mu\nu}$$

$$+ 4\alpha P_{\mu\alpha\nu\beta} \nabla^\alpha \nabla^\beta (e^{-\zeta\phi}) + \alpha e^{-\zeta\phi} c_2 \zeta^2 \left[2(\nabla\phi)^2 \nabla_\mu\phi \nabla_\nu\phi - \frac{1}{2} g_{\mu\nu}(\nabla\phi)^4 \right] = 0$$

$$(11)$$

Here we have (Binétruy *et al.*, 2002; Deruelle and Madore, in press)

$$H_{\mu\nu} = R R_{\mu\nu} - 2R_{\mu\alpha} R_\nu^\alpha - 2R^{\alpha\beta} R_{\mu\alpha\nu\beta} + R_\mu^{abc} R_{\nu abc}$$

$$- \frac{1}{4} g_{\mu\nu} R_{\mathrm{GB}}^2$$

$$P_{\mu\alpha\nu\beta} = R_{\mu\alpha\nu\beta} + R_{\mu\beta} g_{\nu\alpha} + R_{\nu\alpha} g_{\mu\beta} - R_{\mu\nu} g_{\alpha\beta} - R_{\alpha\beta} g_{\mu\nu}$$

$$+ \frac{1}{2} R(g_{\mu\nu} g_{\alpha\beta} - g_{\mu\beta} g_{\nu\alpha}) \qquad (12)$$

Varying with respect to ϕ we obtain

$$2\zeta\nabla^2\phi - \alpha\zeta e^{-\zeta\phi}\left[R_{\rm GB}^2 - 3c_2\zeta^2(\nabla\phi)^4\right] - 2\Lambda_{(5)}V'(\phi)$$

$$-4\alpha c_2\zeta^2 e^{-\zeta\phi}[(\nabla\phi)^2\nabla^2\phi + 2\nabla^\mu\phi\nabla^\nu\phi\nabla_\mu\nabla_\nu\phi] = 0 \qquad (13)$$

Greek indices denote five-dimensional components (0, 1, 2, 3, 5), while Latin three-dimensional.

The (00)-component of the generalized Einstein's equations is

$$3n^2 F + \frac{3n^2}{b^2}\left(-\frac{a'}{a} + \frac{a'b'}{ab} + \frac{\dot{a}b\dot{b}}{an^2}\right) - \frac{2}{3}(\dot\phi)^2 - \frac{2}{3}\frac{n^2}{b^2}(\phi')^2 - n^2\Lambda_{(5)}V(\phi)$$

$$+ 2\alpha e^{-4\phi/3}H_{00} + \frac{16\alpha}{3b^4}P_{0505}\left[-\phi' + \frac{4}{3}(\phi')^2 + \frac{b\dot{b}}{n^2}\dot\phi + \frac{b'}{b}\phi'\right]e^{-4\phi/3}$$

$$+ \frac{16\alpha}{3a^4}P_{0i0j}h^{ij}\left(\frac{a\dot{a}}{n^2}\dot\phi - \frac{aa'}{b^2}\phi'\right)e^{-4\phi/3}$$

$$+ \frac{32\alpha}{27}e^{-4\phi/3}(\nabla\phi)^2\left[\frac{3}{4}\dot\phi^2 + \frac{1}{4}\frac{n^2}{b^2}(\phi')^2\right] = 0 \qquad (14)$$

where we use the conventions

$$F := \frac{1}{a^2}\left(\frac{\dot{a}^2}{n^2} - \frac{(a')^2}{b^2}\right) + \frac{k}{a^2}, \quad (\nabla\phi)^2 = -\frac{1}{n^2}\dot\phi^2 + \frac{1}{b^2}(\phi')^2 \qquad (15)$$

A dot denotes a partial derivative with respect to the time, while a prime denotes derivative with respect to the extra dimension, denoted by y.

The (05)-component of the Einstein's equations is given by

$$-3\left(\frac{\dot{a}'}{a} - \frac{\dot{a}n'}{an} - \frac{a'\dot{b}}{ab}\right) - \frac{4}{3}\dot\phi\phi' + 2\alpha e^{-4\phi/3}H_{05} + \frac{32\alpha}{27}e^{-4\phi/3}(\nabla\phi)^2\dot\phi\phi'$$

$$+ \frac{16\alpha}{3b^2n^2}P_{0505}\left[-\dot\phi' + \frac{4}{3}\dot\phi\phi' + \frac{n'}{n}\dot\phi + \frac{\dot{b}}{b}\phi'\right]e^{-4\phi/3}$$

$$+ \frac{16\alpha}{3a^4}P_{0i5j}h^{ij}\left(\frac{a\dot{a}}{n^2}\dot\phi - \frac{aa'}{b^2}\phi'\right)e^{-4\phi/3} = 0 \qquad (16)$$

The (55)-component is given by

$$-3b^2 F + \frac{3b^2}{n^2}\left(-\frac{\ddot{a}}{a} + \frac{\dot{a}\dot{n}}{an} + \frac{a'nn'}{ab^2}\right) - \frac{2}{3}(\phi')^2 - \frac{2}{3}\frac{b^2}{n^2}(\dot\phi)^2 + b^2\Lambda_{(5)}V(\phi)$$

$$+ 2\alpha e^{-4\phi/3}H_{55} + \frac{16\alpha}{3n^4}P_{0505}\left[-\ddot\phi + \frac{4}{3}(\dot\phi)^2 + \frac{\dot{n}}{n}\dot\phi + \frac{nn'}{b^2}\phi'\right]e^{-4\phi/3}$$

$$+ \frac{16\alpha}{3a^4} P_{5i5j} h^{ij} \left(\frac{a\dot{a}}{n^2} \dot{\phi} - \frac{aa'}{b^2} \phi' \right) e^{-4\phi/3}$$

$$+ \frac{32\alpha}{27} e^{-4\phi/3} (\nabla\phi)^2 \left[\frac{3}{4}(\phi')^2 + \frac{1}{4}\frac{b^2}{n^2}(\dot{\phi})^2 \right] = 0 \tag{17}$$

The (ij)-component of the generalized Einstein's equations is a multiple of h_{ij}. Setting this proportionality term equal to zero gives

$$-2 \left(\frac{a\ddot{a}}{n^2} - \frac{a\dot{a}\dot{n}}{n^3} - \frac{aa'n'}{nb^2} \right) - 2 \left(-\frac{aa'}{b^2} + \frac{aa'b'}{b^3} + \frac{a\dot{a}\dot{b}}{bn^2} \right) - a^2 F$$

$$- \frac{a^2}{n^2} \left(\frac{\ddot{b}}{b} - \frac{nn''}{b^2} + \frac{nn'b'}{b^3} - \frac{\dot{n}\dot{b}}{nb} \right) + \frac{2}{3}a^2(\nabla\phi)^2 + a^2\Lambda_{(5)}V(\phi)$$

$$+ 2\alpha e^{-4\phi/3} \left(\frac{1}{3} h^{ij} H_{ij} \right) - \frac{8\alpha}{27} e^{-4\phi/3} a^2 (\nabla\phi)^4$$

$$+ \frac{16\alpha}{9n^4} P_{i0j0} h^{ij} \left[-\ddot{\phi} + \frac{4}{3}(\dot{\phi})^2 + \frac{\dot{n}}{n}\dot{\phi} + \frac{nn'}{b^2}\phi' \right] e^{-4\phi/3}$$

$$- \frac{32\alpha}{9n^2b^2} P_{0i5j} h^{ij} \left[-\dot{\phi}' + \frac{4}{3}\dot{\phi}\phi' + \frac{n'}{n}\dot{\phi} + \frac{\dot{b}}{b}\phi' \right] e^{-4\phi/3}$$

$$+ \frac{16\alpha}{9b^4} P_{i5j5} h^{ij} \left[-\phi' + \frac{4}{3}(\phi')^2 + \frac{b\dot{b}}{n^2}\dot{\phi} + \frac{b'}{b}\phi' \right] e^{-4\phi/3}$$

$$+ \frac{16\alpha}{9a^4} P_{ikjm} h^{ij} h^{km} \left(\frac{a\dot{a}}{n^2}\dot{\phi} - \frac{aa'}{b^2}\phi' \right) e^{-4\phi/3} = 0 \tag{18}$$

The contraction $h^{ij} H_{ij}$ is given in the Appendix.

3. REDUCTION AND AN EXACT FINE-TUNED SOLUTION

We consider flat spatial sections $(k = 0)$ in Eq. (10) and introduce the following ansatz:

$$a = a(t)A(y), \ n = n(t)N(y), \ b = b(t)B(y),$$

$$\phi = \sigma(t) + \phi(y) \tag{19}$$

where

$$a(t) = a_0 e^{Ht}, \ b(t) = b_0 e^{-2\sigma_1 Ht/3},$$

$$n(t) = n_1 b(t), \quad \sigma(t) = \sigma_1 Ht + \sigma_2 \tag{20}$$

with $H, a_0, b_0, \sigma_1, n_1, \sigma_2$ constants. The field equations become exactly those presented in the Appendix. Now we make the assumption

$$A(y) = A_0 e^{f_1 y}, \quad B(y) = B_0 e^{f_2 y}$$

$$N(y) = N_0 e^{f_2 y}, \quad \phi(y) = -\frac{3}{2} f_2 y + \phi_0 \tag{21}$$

We obtain five equations constraining the numerical parameters of our solution. Inspecting the metric and the resulting equations it is easy to see that we can set $a_0 = 1 = A_0$. It appears, therefore, that there exist eight (8) independent parameters, namely $H, f_1, f_2, N_1 := n_1 N_0, \Phi_0 := \sigma_2 + \phi_0, \sigma_1, b_0,$ and B_0.

The solution for the metric is therefore written

$$ds^2 = -b_0^2 N_1^2 e^{-4\sigma_1 Ht/3} e^{2f_2 y} \, dt^2 + e^{2Ht} e^{2f_1 y} \left[dx_1^2 + dx_2^2 + dx_3^2 \right]$$

$$+ b_0^2 B_0^2 e^{-4\sigma_1 Ht/3} e^{2f_2 y} \, dy^2 \tag{22}$$

However, a closer inspection shows that rescaling the metric and redefining the internal coordinates x_j, we can set $b_0 = 1 = N_1 = B_0$. This also occurs from the five field equations. So there exist five (5) independent constants $H, f_1, f_2, \Phi_0, \sigma_1$ and the metric takes the final form

$$ds^2 = -e^{-4\sigma_1 Ht/3} e^{2f_2 y} \, dt^2 + e^{2Ht} e^{2f_1 y} \left[dx_1^2 + dx_2^2 + dx_3^2 \right]$$

$$+ e^{-4\sigma_1 Ht/3} e^{2f_2 y} \, dy^2 \tag{23}$$

The solution for the scalar field is

$$\phi(t, y) = \sigma_1 Ht - \frac{3}{2} f_2 y + \Phi_0 \tag{24}$$

Inspecting the field equations we see that the following

$$\sigma_1 = 0, \quad f_1 = f_2 := f, \quad H^2 = f^2,$$

$$e^{4\Phi_0/3} = 2\alpha f^2, \quad f^2 = -\frac{4}{3} \Lambda_{(5)},$$

$$\frac{8}{3} \alpha \Lambda_{(5)} = -1 \tag{25}$$

is an *exact* fine-tuned solution, which means that the cosmological constant and the Hubble constant are directly related. This solution though exact does not localize four-dimensional gravity on the brane, located at $y = 0$.

4. CLASSES OF NON–FINE-TUNED SOLUTIONS

We set $f_1 = f_2 = f$ into the five equations of the Appendix and use the metric, Eq. (23). We thus have four constants, namely f, H, Φ_0, σ_1. From the

(05)-equation we obtain

$$f^2 = \frac{2}{9}\left(1 - \frac{8}{3}\sigma_1 - \frac{2}{9}\sigma_1^2\right)H^2 \tag{26}$$

Equation (26) constrains the allowed value of σ_1 so that $-12.36 \simeq -6 - \frac{9\sqrt{2}}{2} \leq \sigma_1 \leq -6 + \frac{9\sqrt{2}}{2} \simeq 0.364$.

We consider now that $|\sigma_1|$ is much smaller than unity. Then we keep the first-order terms in the field equations of the Appendix, with respect to σ_1. We obtain

$$f^2 = \frac{2}{9}\left(1 - \frac{8}{3}\sigma_1\right)H^2 \tag{27}$$

The last term, in the parentheses of Equation (26), contributes 0.02 to the sum only, validating our approximation, for the above range of σ_1. Using Equation (27) into the ϕ-equation, we obtain for the bulk cosmological constant,

$$\Lambda_{(5)}e^{4\Phi_0/3} = -\left(1 + \frac{1}{3}\sigma_1\right)H^2$$
$$+ \alpha e^{-4\Phi_0/3}\left(-\frac{62}{9} + \frac{20}{27}\sigma_1\right)H^4 \tag{28}$$

From the addition of the (00) and (55) components, we obtain

$$(4\sigma_1 + 3) = 2\alpha e^{-4\Phi_0/3}\left(\frac{4}{3} - \frac{16}{9}\sigma_1\right)H^2 \tag{29}$$

From the (00)-equation, using Eq. (28), we get

$$(\sigma_1 + 3) + \alpha e^{-4\Phi_0/3}\left[-\frac{4}{3} + \frac{28}{9}\sigma_1\right]H^2 = 0 \tag{30}$$

Finally the (ij)-equation, with the aid of Eq. (28), gives

$$3(\sigma_1 + 1) + \frac{4}{81}\alpha e^{-4\Phi_0/3}\left[\frac{503}{3}\sigma_1 + 85\right]H^2 = 0 \tag{31}$$

Given the complexity of the field equations, the simplicity of the above, reduced field equations, is quite interesting.

We define $F(\sigma_1) := \alpha e^{-4\Phi_0/3}H^2$. This function assumes the forms F_1, F_2, F_3 as these emerge from Eqs. (29), (30) and Eq. (31), respectively.

Given the extreme complexity of the field equations it is surprising that over the whole range of $-0.3 \leq \sigma_1 \leq 0.2$ the two functions $F_1(\sigma_1)$, $F_2(\sigma_2)$ coincide. Also, in this interval and in $-0.9 \leq \sigma_1 \leq -0.55$ these three functions coincide
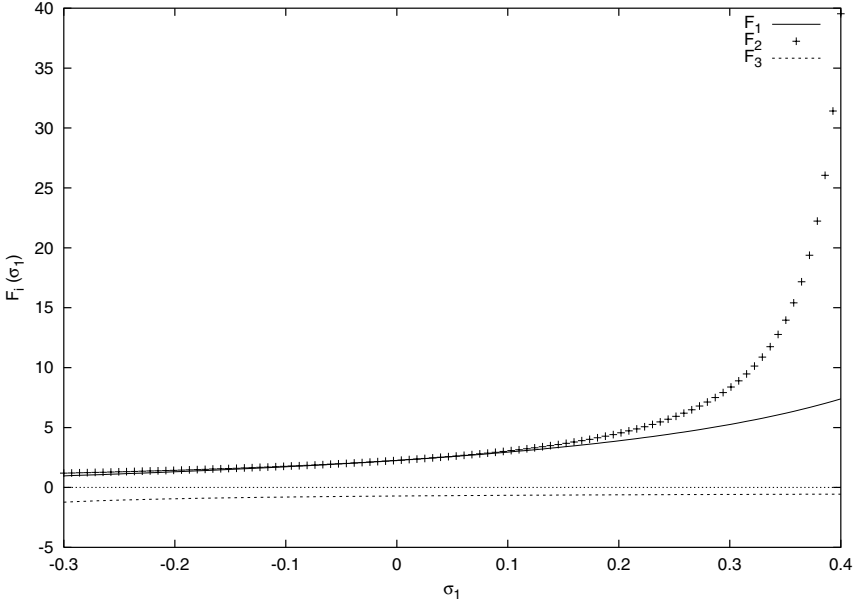
Fig. 1. The functions $F_i(\sigma_1)$ over $-0.3 \leq \sigma_1 \leq 0.4$.

with adequate accuracy as is shown in Figs. 1–2. The difference in their value is suppressed in this intervals, as compared to other intervals.

Our aim here is not to give mathematically exact solutions, but to stress that with acceptable accuracy we can find cosmological models that show many interesting features. We assume therefore, given also the approximation we have made for small $|\sigma_1|$, that three of the five field equations, namely the $(00)+(55)$, (00), and (ij) components, give

$$F(\sigma_1) := \alpha e^{-4\Phi_0/3} H^2 \simeq F_1(\sigma_1) = \frac{3 + 4\sigma_1}{\left(\frac{4}{3} - \frac{16}{9}\sigma_1\right)} \tag{32}$$

where $\sigma_1 \in (-0.9, -0.55) \cup (-0.3, 0.2)$.

Therefore, the Hubble constant in string units is given in terms of the initial value of the scalar field as

$$\alpha H^2 = e^{4\Phi_0/3} \frac{3 + 4\sigma_1}{\left(\frac{4}{3} - \frac{16}{9}\sigma_1\right)} \tag{33}$$

and so f^2, which determines the spatial dependence of the scalar field through Equation (24), is determined through Eq. (27).
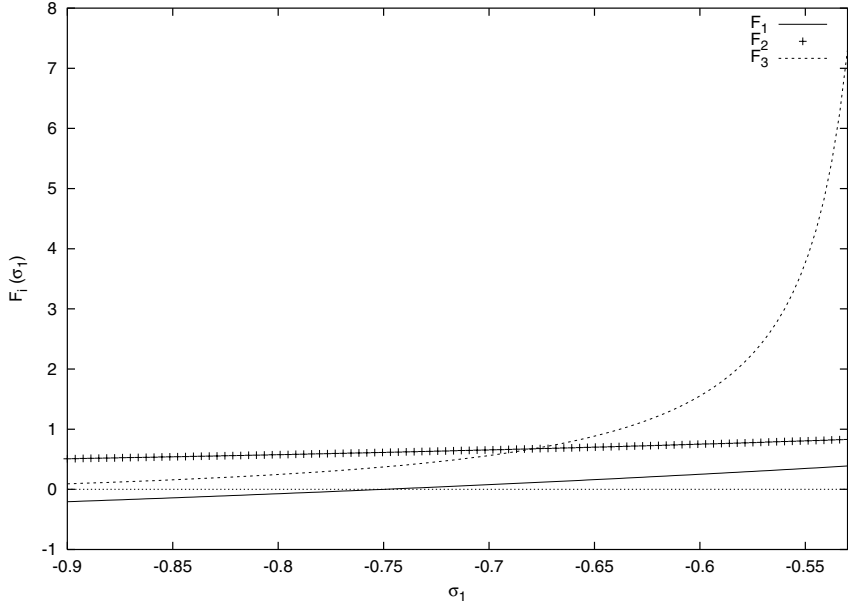
Fig. 2. The functions $F_i(\sigma_1)$ over $-0.9 \leq \sigma_1 \leq -0.5$.

Multiplying Eq. (28) by α, and using Eq. (33), we obtain the bulk cosmological constant in string units,

$$\alpha\Lambda_{(5)} = \frac{(3 + 4\sigma_1)}{\left(\frac{4}{3} - \frac{16}{9}\sigma_1\right)^2} \left[-22 - 24\sigma_1 + \frac{32}{9}\sigma_1^2\right] \tag{34}$$

The dependence of the bulk comological constant on σ_1 is shown in Fig. 3.

Thus we do not have to fine-tune the Hubble constant or the bulk cosmological constant $\Lambda_{(5)}$. These, as well the constant f, are specified in terms of σ_1, which controls the temporal evolution of the scalar field through Eq. (24), and the initial value Φ_0. So σ_1 determines the bulk cosmological constant through Eq. (34), but the Hubble constant H is determined by the initial value of the scalar field as well, (Eq. (33)) and can be freely adjusted. The metric is given by Eq. (23) (with $f_1 = f_2 = f$) and the scalar field by Eq. (24).

Finally if we assume that $f_1 \neq f_2$, there will, in general, exist *exact* solutions of the five field equations, for the five independent constants f_1, f_2, H, σ_1, Φ_0. These will also be determined in terms of $\Lambda_{(5)}$. In order to determine them one has to resort to numerical methods and this will be the subject of a future work.
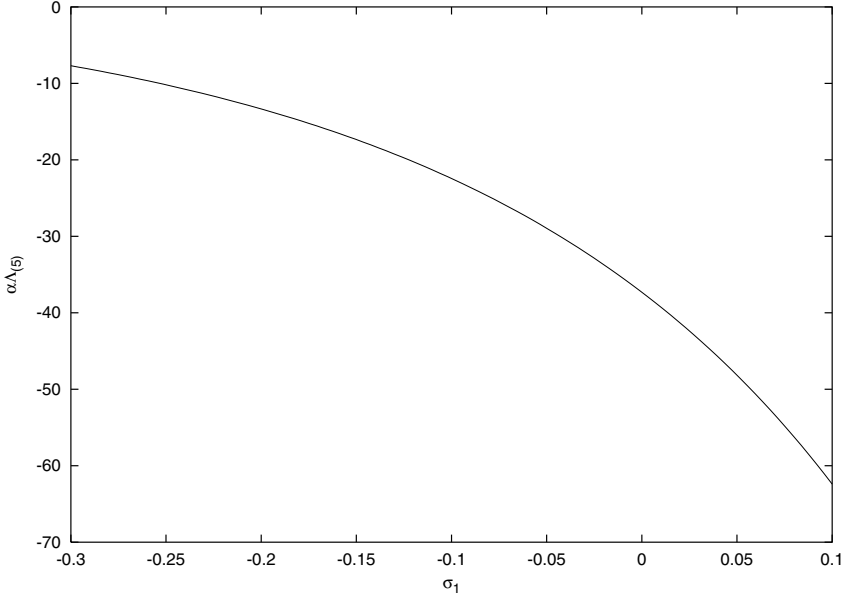
Fig. 3. The bulk cosmological constant over $-0.3 \leq \sigma_1 \leq 0.1$.

5. LOCALIZATION OF GRAVITY

The four-dimensional scalar curvature, with flat spatial sections for the metric of Equation (10), is given by $^{(4)}R = (\frac{6}{n^2})(\frac{\ddot{a}}{a} - \frac{\dot{a}\dot{n}}{an} + (\frac{\dot{a}}{a})^2)$. If one substitutes the five-dimensional scalar curvature R and the Gauss–Bonnet contribution into the action functional, then the integrated coefficient of the four-dimensional scalar curvature will give the effective four-dimensional Planck mass as perceived by a four-dimensional observer, located on the brane (Binétruy *et al.*, 2003; Mavromatos and Rizos, 2000, 2003). Doing this in a careful manner we obtain

$$M_{\text{Pl}}^2 = M_s^3 \int_0^R dy b \left[1 + \frac{8\alpha e^{-4\phi/3}}{b^2} \left(-\frac{a'}{a} + \frac{a'b'}{ab} + \frac{\dot{a}b\dot{b}}{an^2} \right) \right]$$

$$= M_s^3 e^{-2\sigma_1 Ht/3} \frac{1}{|f|} [1 - e^{-|f|R}] \left(1 - \frac{16}{3}\sigma_1 \frac{3 + 4\sigma_1}{\frac{4}{3} - \frac{16}{9}\sigma_1} \right) \tag{35}$$

This must be finite as $R \to +\infty$, so this is why we have chosen the $f < 0$ solution of Eq. (27). For the negative y-direction we can choose $f > 0$ and match the two bulk solutions continuously on the brane.

The quantity in the large parentheses of Eq. (35) is positive for the range of $\sigma_1 \in (-0.9, -0.55) \cup (-0.3, 0.1)$, giving an overall positive Planck mass. So these

cosmological solutions can localize four-dimensional gravity. Also, we observe that we naturally obtain a time-varying gravitational constant. This occurs due to the nontrivial, different time dependences of the various metric components of the metric, Eq. (23), and also due to the fact that we have the exponential $e^{4\Phi_0/3}$ in Eq. (33). The last comes from the coefficient of the GB term of the action, Eq. (5).

6. DISCUSSION

We have considered the action that results from a consistent truncation to five dimensions of the heterotic string. This action has higher-order gravitational corrections of the form of the Gauss–Bonnet term. However this term enters the expansion, multiplied not only by the string constant but also by the exponential of the scalar field. This makes the resulting field equations very complicated, compared to the usual case when the exponential term is absent. Due to this fact, only solutions with the four-dimensional Poincare-invariant form of Eq. (8), have been considered in the literature. In this paper we showed that there exist one exact and a family of approximate solutions, continuously dependent on the parameter σ_1, with the metric given by Eq. (23).

The important features of these cosmological models can be summarized as follows: The metric cannot be factored as in a simple, conformally flat case, since in general $\sigma_1 \neq -(3/2)$. The temporal and spatial dependence of the metric components is nontrivial and does not allow a conformally flat solution even in the four-dimensional subcase. The metric cannot be simplified any further by a coordinate transformation. Also the brane, situated at $y = 0$, can localize four-dimensional gravity. This is due to the fact that the four-dimensional effective Planck mass is finite and positive for the allowed range of the parameter σ_1.

More importantly the parameters of the theory need not be fine-tuned. By this we mean that the parameters of our action such as the bulk cosmological constant need not be fine-tuned to a specific value in order to obtain a desired solution. The bulk cosmological constant, is directly related to the temporal evolution of the scalar field, i.e., to σ_1. The Hubble constant, however, is freely determined from the initial value of the scalar field and so a proper choice of the last can accomodate observational data.

Finally the four-dimensional gravitational constant varies with time and follows the exponential expansion of the four-dimensional scale factor (Antoniadis *et al*., 1998; Arkadi-Hamed *et al*., 1998; Uzan, 2003).

The action used in this paper is a realistic one because it occurs in a consistent way from heterotic string theory (Gross and Sloan, 1987). So it is important to search for brane cosmological solutions that give realistic four-dimensional cosmological models. Because the field equations are complicated, the use of combined numerical and analytical techniques is necessary. One can numerically

search for solutions with the metric assuming the form of Equation (23) and without any other assumption. Work along these lines is in progress.

APPENDIX

Contracting the first of Eqs. (12) we get

$$h^{ij} H_{ij} = -\frac{1}{4} a^2 R_{\mathrm{GB}}^2 + \frac{a^2}{n^2} H_{00} - \frac{a^2}{b^2} H_{55}. \qquad (A.1)$$

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